Mathematics: analysis and approaches

Higher level - Paper 3 ■ **Worked Solutions v1** ■

1. [Maximum mark: 29]

(a) $\mathrm{E}(X)=1\cdot\frac{1}{6}+2\cdot\frac{2}{6}+5\cdot\frac{2}{6}+7\cdot\frac{1}{6}=\frac{1}{6}+\frac{4}{6}+\frac{10}{6}+\frac{7}{6}=\frac{22}{6} \Rightarrow \mathrm{E}(X)=\frac{11}{3}\approx 3.66$ [2]

(b) (i) 3 ways for sum of numbers = 5: $1+2+2$, $2+1+2$ and $2+2+1$ [3]

(ii) $\mathrm{P}(\text{sum}=5)=3\cdot\frac{1}{6}\cdot\frac{1}{3}\cdot\frac{1}{3}=\frac{1}{18}\approx 0.0556$ [3]

(iii) 6 ways for sum = 9: $2+2+5, 2+5+2, 5+2+2;\ 1+1+7, 1+7+1, 7+1+1$

$\mathrm{P}(\text{sum}=3)=\left(3\cdot\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}\right)+\left(3\cdot\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}\right)=\frac{1}{9}+\frac{3}{216}=\frac{1}{8}=0.125$ [3]

(iv) 4 ways for median = 1: 1-1-1, 1-1-2, 1-1-5 or 1-1-7

$\mathrm{P}(\text{median}=1)=\left(\frac{1}{6}\right)^3+\left(\frac{1}{6}\right)^2\left(\frac{1}{3}\right)+\left(\frac{1}{6}\right)^2\left(\frac{1}{3}\right)+\left(\frac{1}{6}\right)^3=\frac{1}{36}\approx 0.0278$ [3]

(c) let random variable X represent the # of 5s (binomial probability); $X\sim \mathrm{B}\left(10,\frac{1}{3}\right)$

$\mathrm{P}(X<4)=\mathrm{P}(X\le 3)\approx 0.559$ [3]

(d) (i) $\mathrm{P}(X\ge 1)=1-\mathrm{P}(X=0)=1-\left(\frac{2}{3}\right)^n>0.95 \Rightarrow \left(\frac{2}{3}\right)^n<0.05$

$\left(\frac{2}{3}\right)^n=0.05 \Rightarrow n\approx 7.38838... \Rightarrow n=8$ **Q.E.D.** [2]

(ii) $\mathrm{P}(X\ge 1)=1-\mathrm{P}(X=0)=1-\left(\frac{5}{6}\right)^n>0.85 \Rightarrow \left(\frac{5}{6}\right)^n<0.15$

$\left(\frac{5}{6}\right)^n=0.15 \Rightarrow n\approx 10.405... \Rightarrow n=11$ [3]

(e) binomial probability: $\mathrm{E}(X)=np$ and $\mathrm{Var}(X)=np(1-p)$

$X=1$: $4.8=8p \Rightarrow p=0.6 \Rightarrow \mathrm{P}(X=1)=0.6$

$X=2$: $1.5=8p(1-p) \Rightarrow 3=16p(1-p) \Rightarrow 16p^2-16p+3=0$

$p=0.25$ or $p=0.75$; $p\ne 0.75$ since $\mathrm{P}(X=1)+\mathrm{P}(X=2)>1$; hence, $\mathrm{P}(X=2)=0.25$

$W=\mathrm{P}(X=1\text{ or }2)=\mathrm{P}(x=1)+\mathrm{P}(x=2)=0.6+0.25$ Thus, $W=0.85$ [7]

2. [Maximum mark: 26]

(a) p = cost (\$/meter) of pipeline from X to B
$6p$ = cost (\$/meter) of pipeline from A to X

$\text{XB} = 1000 - x$ meters; $\text{AX} = \sqrt{x^2 + 500^2} = \sqrt{x^2 + 250000}$ meters

Thus, total cost of AX+XB $= C = 6p\sqrt{x^2 + 250000} + p(1000 - x)$

$$C = 6p\sqrt{x^2 + 250000} + 1000p - px$$ **Q.E.D.** [2]

(b) (i) note that p is a constant

$$\frac{dC}{dx} = \frac{d}{dx}\left(6p\sqrt{x^2 + 250000} + 1000p - px\right) = \frac{d}{dx}\left(6p\left(x^2 + 250000\right)^{\frac{1}{2}} + 1000p - px\right)$$

$$= 6p\left(\frac{1}{2}\left(x^2 + 250000\right)^{-\frac{1}{2}}(2x)\right) + 0 - p$$

$$= \frac{6px}{\sqrt{x^2 + 250000}} - p = \frac{6px}{\sqrt{x^2 + 250000}} - \frac{p\sqrt{x^2 + 250000}}{\sqrt{x^2 + 250000}}$$

$$\frac{dC}{dx} = \frac{6px - p\sqrt{x^2 + 250000}}{\sqrt{x^2 + 250000}}$$

(ii) $\frac{dC}{dx} = 0 \Rightarrow 6px - p\sqrt{x^2 + 250000} = 0 \Rightarrow 6x - \sqrt{x^2 + 250000} = 0$

$x \approx 84.5154...$

Evaluate $\frac{dC}{dx}$ for values of x to left and right of $x \approx 84.5154...$ to justify C is a minimum

For $x = 80$: $\frac{dC}{dx} \approx (-0.05206...)p < 0 \Rightarrow C$ is decreasing [<u>note</u>: p must be positive]

For $x = 90$: $\frac{dC}{dx} \approx (0.06292...)p > 0 \Rightarrow C$ is increasing

Thus, C must have a minimum value when $x \approx 84.5$ meters [7]

(c) $C(84.5154...) \approx (3958.0399...)p$

Thus, to an accuracy of 3 significant figures, the minimum total cost is ***3960p*** [1]

(d) Let angle AXE $= \alpha$; then $\alpha + \theta = 180°$

$\tan\alpha = \frac{500}{x} \Rightarrow \alpha = \tan^{-1}\left(\frac{500}{84.5154...}\right) \approx 80.40593...°$

$\theta = 180° - 80.40593...° \Rightarrow \theta \approx 99.594...°$ Thus, $\theta \approx 99.6°$ (3 significant figures) [2]

[*continued on next page*]

(e) (i) $\theta = 120° \Rightarrow \alpha = 60°$

$\tan 60° = \dfrac{500}{x} \Rightarrow x = \dfrac{500}{\tan 60°} \approx 288.675...$ Thus, $x \approx 289$ meters (3 significant figures)

(ii) $C(288.675...) \approx (4175.4265...)\,p$

increase $\approx (4175.4265...)\,p - (3958.0399...)\,p \approx (217.3866...)\,p$

% increase $\approx 100\dfrac{(217.3866...)\,p}{(3958.0399...)\,p} \approx 5.49\%$ [4]

(f) total cost: $C = 6p\sqrt{x^2+a^2} + pb - px$

$$\frac{dC}{dx} = \frac{d}{dx}\left(6p\left(x^2+a^2\right)^{\frac{1}{2}} + pb - px\right) = 6p\cdot\frac{1}{2}\left(x^2+a^2\right)^{-\frac{1}{2}}2x + 0 - p$$

$$= \frac{6px}{\sqrt{x^2+a^2}} - p = \frac{6px}{\sqrt{x^2+a^2}} - \frac{p\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}} = \frac{6px - p\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}} = 0$$

$6px - p\sqrt{x^2+a^2} = 0 \Rightarrow p\sqrt{x^2+a^2} = 6px \Rightarrow \sqrt{x^2+a^2} = 6x$

$\left(\sqrt{x^2+a^2}\right)^2 = (6x)^2$ squaring both sides may lead to extraneous solution(s); so, check solution(s)

$x^2 + a^2 = 36x^2 \Rightarrow 35x^2 = a^2 \Rightarrow x = \pm\dfrac{a}{\sqrt{35}}$; $x > 0$, so $x = -\dfrac{a}{\sqrt{35}}$ is an extraneous solution

When $a = 500$, then $x = \dfrac{500}{\sqrt{35}} \approx 84.5154...$ which agrees with result from part (b) (ii)

Thus, total cost is a minimum when $x = \dfrac{a}{\sqrt{35}}$ $\left[\text{OR } x = \dfrac{a\sqrt{35}}{35}\right]$ [6]

(g) $\tan\alpha = \dfrac{a}{x} \Rightarrow \alpha = \tan^{-1}\left(\dfrac{a}{a/\sqrt{35}}\right) = \tan^{-1}\left(\sqrt{35}\right) \approx 80.40593...°$

$\theta = 180° - 80.40593...° \Rightarrow \theta \approx 99.594...°$ Thus, $\theta \approx 99.6°$ (3 significant figures) [2]

(h) The value of *b* has no effect on the value of *x* that gives a minimum total cost.
There is a direct relationship between *a* and the value of *x* that gives a minimum total cost.
As *a* increases, *x* increases – and vice versa. [2]